

OPTIMAL INVESTMENT STRATEGY FOR RISKY ASSETS

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We design an optimal strategy for investment in a portfolio of assets subject to a multiplicative Brownian motion. The strategy provides the maximal typical long-term growth rate of investor's capital. We determine the optimal fraction of capital that an investor should keep in risky assets as well as weights of different assets in an optimal portfolio. In this approach both average return and volatility of an asset are relevant indicators determining its optimal weight. Our results are particularly relevant for very risky assets when traditional continuous-time Gaussian portfolio theories are no longer applicable.

1. Introduction

The simplest version of the problem we are going to address in this manuscript is rather easy to formulate. Imagine that you are an investor with some starting capital, which you can invest in just one risky asset. You decided to use the following simple strategy: you always maintain a given fraction $0 < r < 1$ of your total current capital invested in this asset, while the rest (given by the fraction $1 - r$) you wisely keep in cash. You select a unit of time (say a week, a month, a quarter, or a year, depending on how closely you follow your investment, and what transaction costs are involved) at which you check the asset's current price, and sell or buy some shares of this asset. By this transaction you adjust the current money equivalent of your investment to the above pre-selected fraction of your total capital.

The question we are interested in is: which investment fraction provides the optimal *typical* long-term growth rate of investor's capital? By typical we mean that this growth rate occurs at large-time horizon in majority of realizations of the multiplicative process. By extending time-horizon one can make this rate to occur with probability arbitrary close to one. Contrary to the traditional economics approach, where the expectation value of an artificial "utility function" of an investor is optimized, the optimization of a typical growth rate does not contain any ambiguity.

In this work we also assume that during the timescale, at which the investor checks and readjusts his asset's capital to the selected investment fraction, the asset's price changes by a random factor, drawn from some probability distribution, and uncorrelated from price dynamics at earlier intervals. In other words, the price of an asset experiences a *multiplicative* random walk with some known probability distribution of steps. This assumption is known to hold in real financial markets beyond a certain time scale [1, 2]. Contrary to continuum theories popular among economists [3] our approach is not limited to Gaussian distributed returns: indeed, we were able to formulate our strategy for a general probability distribution of returns per capital (elementary steps of the multiplicative random walk).

Our purpose here is to illustrate the essential framework through simplest examples. Thus risk-free interest rate, asset's dividends, and transaction costs are ignored (when volatility is large they are indeed negligible). However, the task of including these effects in our formalism is rather straightforward.

The quest of finding a strategy, which optimizes the long-term growth rate of the capital is by no means new: indeed it was first discussed by Daniel Bernoulli in about 1730 in connection with the St. Petersburg game [4]. In the early days of information sciences, Shannon [5] had considered the application of the concept of information entropy in designing optimal strategies in such games as gambling. Working from the foundations of Shannon, Kelly has specifically designed an optimal gambling strategy in placing bets [6], when a gambler has some incomplete information about the winning outcome (a “noisy information channel”). In modern day finance, especially the investment in very risky assets is no different from gambling. The point Shannon and Kelly wanted to make is that, given that the odds are slightly in your *favor* albeit with large uncertainty, the gambler should not bet his whole capital at every time step. On the other hand, he would achieve the biggest long-term capital growth by betting some specially optimized fraction of his whole capital in every game. This cautious approach to investment is recommended in situations when the volatility is very large. For instance, in many emergent markets the volatility is huge, but they are still swarming with investors, since the long-term return rate in some cautious investment strategy is favorable.

Later on Kelly's approach was expanded and generalized in the works of Breiman [8]. Our results for multi-asset optimal investment are in agreement with his exact but non-constructive equations. In some special cases, Merton and Samuelson [3] have considered the problem of portfolio optimization, when the underlying asset is subject to a multiplicative *continuous* Brownian motion with Gaussian price fluctuations. Overall, we feel that the topic of optimal long-term investment has not been adequately exploited, and many interesting consequences are yet to be revealed.

The plan of this paper is as follows: in Sec. 2 we determine the optimal investment fraction in an (unrealistic) situation when an investor is allowed to invest in only one risky asset. Section 3 generalizes these results for a more realistic case when an investor can keep his capital in a multi-asset portfolio. In this case we determine the optimal weights of different assets in this portfolio.

2. Optimal Investment Fraction for One Asset

We first consider a situation, when an investor can spend a fraction of his capital to buy shares of just one risky asset. The rest of his money he keeps in cash. Generalizing Kelly [6], we consider the following simple strategy of the investor: he regularly checks the asset's current price $p(t)$, and sells or buys some asset shares in order to keep the current market value of his asset holdings a pre-selected fraction r of his total capital. These readjustments are made periodically at a fixed interval, which we refer to as readjustment interval, and select it as the discrete unit of time. In this work the readjustment time interval is selected once and for all, and we do not attempt optimization of its length.

We also assume that on the time-scale of this readjustment interval the asset price $p(t)$ undergoes a geometric Brownian motion:

$$p(t+1) = e^{\eta(t)} p(t), \quad (2.1)$$

i.e. at each time step the random number $\eta(t)$ is drawn from some probability distribution $\pi(\eta)$, and is independent of its value at previous time steps. This exponential notation is particularly convenient for working with multiplicative noise, keeping the necessary algebra at minimum. Under these rules of dynamics the logarithm of the asset's price, $\ln p(t)$, performs a random walk with an average drift $v = \langle \eta \rangle$ and a dispersion $D = \langle \eta^2 \rangle - \langle \eta \rangle^2$.

It is easy to derive the time evolution of the total capital $W(t)$ of an investor, following the above strategy:

$$W(t+1) = (1-r)W(t) + rW(t)e^{\eta(t)}. \quad (2.2)$$

Let us assume that the value of the investor's capital at $t = 0$ is $W(0) = 1$. The evolution of the expectation value of the total capital $\langle W(t) \rangle$ after t time steps is obviously given by the recursion $\langle W(t+1) \rangle = (1-r + r\langle e^\eta \rangle)\langle W(t) \rangle$. When $\langle e^\eta \rangle > 1$, at first thought the investor should invest all his money in the risky asset. Then the expectation value of his capital would enjoy an exponential growth with the fastest growth rate. However, it would be totally unreasonable to expect that in a *typical* realization of price fluctuations, the investor would be able to attain the *average* growth rate determined as $v_{\text{avg}} = d\langle W(t) \rangle / dt$. This is because the main contribution to the expectation value $\langle W(t) \rangle$ comes from exponentially unlikely outcomes, when the price of the asset after a long series of favorable events with $\eta > \langle \eta \rangle$ becomes exponentially big. Such outcomes lie well beyond reasonable fluctuations of $W(t)$, determined by the standard deviation \sqrt{Dt} of $\ln W(t)$ around its average value $\langle \ln W(t) \rangle = \langle \eta \rangle t$. For the investor who deals with just one realization of the multiplicative process it is better not to rely on such unlikely events, and maximize his gain in a *typical* outcome of a process. To quantify the intuitively clear concept of a *typical* value of a random variable x , we define x_{typ} as a median [9] of its distribution, i.e. x_{typ} has the property that $\text{Prob}(x > x_{\text{typ}}) = \text{Prob}(x < x_{\text{typ}}) = 1/2$. In a multiplicative process (2.2) with

$r = 1$, $W(t+1) = e^{\eta(t)}W(t)$, one can show that $W_{\text{typ}}(t)$ — the typical value of $W(t)$ — grows exponentially in time: $W_{\text{typ}}(t) = e^{\langle \eta \rangle t}$ at a rate $v_{\text{typ}} = \langle \eta \rangle$, while the expectation value $\langle W(t) \rangle$ also grows exponentially as $\langle W(t) \rangle = \langle e^\eta \rangle^t$, but at a faster rate given by $v_{\text{avg}} = \ln \langle e^\eta \rangle$. Notice that $\langle \ln W(t) \rangle$ always grows with the *typical* growth rate, since those very rare outcomes when $W(t)$ is exponentially big, do not make significant contribution to this average.

The question we are going to address is: which investment fraction r provides the investor with the best *typical* growth rate v_{typ} of his capital. Kelly [6] has answered this question for a particular realization of multiplicative stochastic process, where the capital is multiplied by 2 with probability $q > 1/2$, and by 0 with probability $p = 1 - q$. This case is realized in a gambling game, where betting on the right outcome pays 2:1, while you know the right outcome with probability $q > 1/2$. In our notation this case corresponds to η being equal to $\ln 2$ with probability q and $-\infty$ otherwise. The player's capital in Kelly's model with $r = 1$ enjoys the growth of expectation value $\langle W(t) \rangle$ at a rate $v_{\text{avg}} = \ln 2q > 0$. In this case it is however particularly clear that one should not use maximization of the expectation value of the capital as the optimum criterion. If the player indeed bets all of his capital at every time step, sooner or later he will lose everything and would not be able to continue to play. In other words, $r = 1$ corresponds to the worst *typical* growth of the capital: asymptotically the player will be bankrupt with probability 1. In this example it is also very transparent, where the positive *average* growth rate comes from: after T rounds of the game, in a very unlikely ($\text{Prob} = q^T$) event that the capital was multiplied by 2 at all times (the gambler guessed right all the time!), the capital is equal to 2^T . This exponentially large value of the capital outweighs exponentially small probability of this event, and gives rise to an exponentially growing average. This would offer condolence to a gambler who lost everything.

In this chapter we generalize Kelly's arguments for arbitrary distribution $\pi(\eta)$. As we will see this generalization reveals some hidden results, not realized in Kelly's "betting" game. In a previous publication [7], Galluccio and one of us have already considered generalizing Kelly's theory, where an explicit solution is found for a two-asset model. There it was realised that the typical return and the Lyapunov exponent are equivalent. However, in our present study, we examine under what conditions the Kelly optimization is possible and how to proceed with the general multi-asset case. As we learned above, the growth of the *typical* value of $W(t)$, is given by the drift of $\langle \ln W(t) \rangle = v_{\text{typ}}t$, which in our case can be written as

$$v_{\text{typ}}(r) = \int d\eta \pi(\eta) \ln(1 + r(e^\eta - 1)). \quad (2.3)$$

One can check that $v_{\text{typ}}(0) = 0$, since in this case the whole capital is in the form of cash and does not change in time. In another limit one has $v_{\text{typ}}(1) = \langle \eta \rangle$, since in this case the whole capital is invested in the asset and enjoys its typical growth rate ($\langle \eta \rangle = -\infty$ for Kelly's case). Can one do better by selecting $0 < r < 1$? To find the maximum of $v_{\text{typ}}(r)$ one differentiates (2.3) with respect to r and looks for

a solution of the resulting equation: $0 = v'_{\text{typ}}(r) = \int d\eta \pi(\eta) (e^\eta - 1)/(1 + r(e^\eta - 1))$ in the interval $0 \leq r \leq 1$. If such a solution exists, it is unique since $v''_{\text{typ}}(r) = -\int d\eta \pi(\eta) (e^\eta - 1)^2/(1 + r(e^\eta - 1))^2 < 0$ everywhere. The values of the $v'_{\text{typ}}(r)$ at 0 and 1 are given by $v'_{\text{typ}}(0) = \langle e^\eta \rangle - 1$, and $v'_{\text{typ}}(1) = 1 - \langle e^{-\eta} \rangle$. One has to consider three possibilities:

- (1) $\langle e^\eta \rangle < 1$. In this case $v'_{\text{typ}}(0) < 0$. Since $v''_{\text{typ}}(r) < 0$, the maximum of $v_{\text{typ}}(r)$ is realized at $r = 0$ and is equal to 0. In other words, one should never invest in an asset with negative average return per capital $\langle e^\eta \rangle - 1 < 0$.
- (2) $\langle e^\eta \rangle > 1$, and $\langle e^{-\eta} \rangle > 1$. In this case $v'_{\text{typ}}(0) > 0$, but $v'_{\text{typ}}(1) < 0$ and the maximum of $v(r)$ is realized at some $0 < r < 1$, which is a unique solution to $v'_{\text{typ}}(r) = 0$. The typical growth rate in this case is always positive (because you could have always selected $r = 0$ to make it zero), but not as big as the average rate $\ln \langle e^\eta \rangle$, which serves as an unattainable ideal limit. An intuitive understanding of why one should select $r < 1$ in this case comes from the following observation: the condition $\langle e^{-\eta} \rangle > 1$ makes $\langle 1/p(t) \rangle$ to grow exponentially in time. Such an exponential growth indicates that the outcomes with very small $p(t)$ are feasible and give dominant contribution to $\langle 1/p(t) \rangle$. This is an indicator that the asset price is unstable and one should not trust his whole capital to such a risky investment.
- (3) $\langle e^\eta \rangle > 1$, and $\langle e^{-\eta} \rangle < 1$. This is a safe asset and one can invest his whole capital in it. The maximum $v_{\text{typ}}(r)$ is achieved at $r = 1$ and is equal to $v_{\text{typ}}(1) = \ln \langle \eta \rangle$. A simple example of this type of asset is one in which the price $p(t)$ with equal probabilities is multiplied by 2 or by $a = 2/3$. As one can see this is a marginal case in which $\langle 1/p(t) \rangle = \text{const}$. For $a < 2/3$ one should invest only a fraction $r < 1$ of his capital in the asset, while for $a \geq 2/3$ the whole sum could be trusted to it. The specialty of the case with $a = 2/3$ cannot be guessed by just looking at the typical and average growth rates of the asset! One has to go and calculate $\langle e^{-\eta} \rangle$ to check if $\langle 1/p(t) \rangle$ diverges. This “reliable” type of asset is a new feature of the model with a general $\pi(\eta)$. It is never realized in Kelly’s original model, which always has $\langle \eta \rangle = -\infty$, so that it never makes sense to gamble the whole capital every time.

An interesting and somewhat counterintuitive consequence of the above results is that under certain conditions one can make his capital grow by investing in asset with a negative typical growth rate $\langle \eta \rangle < 0$. Such asset certainly loses value, and its *typical* price experiences an exponential decay. Any investor bold enough to trust his whole capital in such an asset is losing money with the same rate. But as long as the fluctuations are strong enough to maintain a positive *average* return per capital $\langle e^\eta \rangle - 1 > 0$ one can maintain a certain fraction of his total capital invested in this asset and almost certainly make money! A simple example of such mind-boggling situation is given by a random multiplicative process in which the price of the asset with equal probabilities is doubled (goes up by 100%) or divided by 3 (goes down by 66.7%). The typical price of this asset drifts down by 18%

each time step. Indeed, after T time steps one could reasonably expect the price of this asset to be $p_{\text{typ}}(T) = 2^{T/2} 3^{-T/2} = (\sqrt{2/3})^T \simeq 0.82^T$. On the other hand, the average $\langle p(t) \rangle$ enjoys a 17% growth $\langle p(t+1) \rangle = 7/6 \langle p(t) \rangle \simeq 1.17 \langle W(t) \rangle$. As one can easily see, the optimum of the typical growth rate is achieved by maintaining a fraction $r = 1/4$ of the capital invested in this asset. The typical rate in this case is a meager $\sqrt{25/24} \simeq 1.02$, meaning that in the long run one almost certainly gets a 2% return per time step, but it is certainly better than losing 18% by investing the whole capital in this asset.

The temporal evolution of another example is shown in the Fig. 1, where a risky asset varies daily by +30% or -24.4% with equal chance, this is not unlike *daily* variation of some “red chips” quoted in Hong Kong or some Russian companies quoted on the Moscow Stock Exchange. In this example, the stock is almost certainly doomed: in the realization shown on Fig. 1 in four years the price of one share went down by a factor 500, it was practically wiped out. At the same time the investor maintaining the optimal $\simeq 38\%$ investment fraction profited handsomely, making more than 500 times of his starting capital! It is all the more remarkable that this profit is achieved without any insider information but only by dynamically

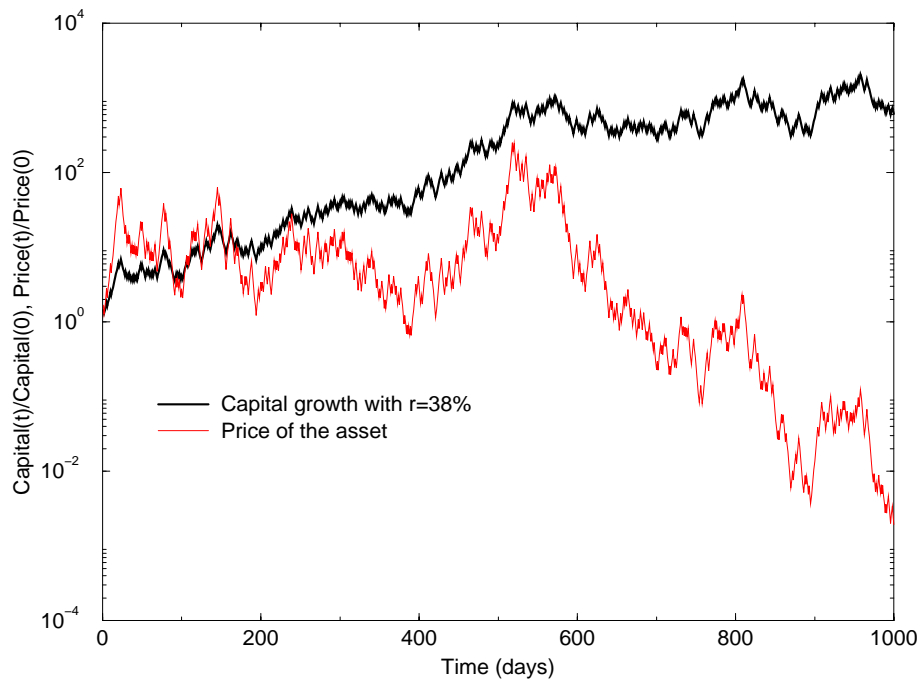


Fig. 1. Temporal evolution of the stock and the optimizing investor's capital. The time units can be interpreted as days and the total period (1000 days) is about 4 years. During this period the doomed stock performed very badly, whereas our investor made huge profit from investing in it dynamically with $r \simeq 38\%$. Not only the optimal strategy performs better, it also has much less volatility. Clearly if one adopts the “buy-and-hold” strategy the investment would vanish together with the company.

managing his investment in such a bad stock. It is unrealistic in this hypothetical case, since we assume that the investment is negligibly small compared to the company's capitalisation so that the investor's impact can be ignored. This would no longer be the case if the stock goes dramatically down and investor's profit dramatically goes up. Somewhere down the road the investor will find himself one of the major players of that doomed stock, market impact then prevents him from ribbing off further profit and he has to either quit for another target or vanish together.

Of course the properties of a typical realization of a random multiplicative process are not fully characterized by the drift $v_{\text{typ}}(r)t$ in the position of the center of mass of $P(h, t)$, where $h(t) = \ln W(t)$ is a logarithm of the wealth of the investor. Indeed, asymptotically $P(h, t)$ has a Gaussian shape $P(h, t) = \frac{1}{\sqrt{2\pi D(r)t}} \exp(-\frac{(h-v_{\text{typ}}(r)t)^2}{2D(r)t})$, where $v_{\text{typ}}(r)$ is given by Eq. (2.3). One needs to know the dispersion $D(r)$ to estimate $\sqrt{D(r)t}$, which is the magnitude of characteristic deviations of $h(t)$ away from its typical value $h_{\text{typ}}(t) = v_{\text{typ}}t$. At the infinite time horizon $t \rightarrow \infty$, the process with the biggest $v_{\text{typ}}(r)$ will certainly be preferable over any other process. This is because the separation between typical values of $h(t)$ for two different investment fractions r grows linearly in time, while the span of typical fluctuations grows only as a \sqrt{t} . However, at a finite time horizon the investor should take into account both $v_{\text{typ}}(r)$ and $D(r)$ and decide what he prefers: moderate growth with small fluctuations or faster growth with still bigger fluctuations. To quantify this decision one needs to introduce an investor's "utility function" which we will not attempt in this work. The most conservative players are advised to always keep their capital in cash, since with any other arrangement the fluctuations will certainly be bigger. As a rule one can show that the dispersion $D(r) = \int \pi(\eta) \ln^2[1 + r(e^\eta - 1)] d\eta - v_{\text{typ}}^2$ monotonically increases with r . Therefore, among two solutions with equal $v_{\text{typ}}(r)$ one should always select the one with a smaller r , since it would guarantee smaller fluctuations.

We proceed with deriving analytic results for the optimal investment fraction r in a situation when fluctuations of asset price during one readjustment period (one step of the discrete dynamics) are small. This approximation is usually justified for developed markets, if the investor sells and buys asset to maintain his optimal ratio on let's say monthly basis. Indeed, the month to month fluctuations in, for example, Dow-Jones Industrial Average i) to a good approximation are uncorrelated random numbers; ii) seldom rise above few percent, so that the assumption that $\eta(t) \ll 1$ is justified.

Here it is more convenient to switch to the standard notation. It is customary to use the random variable

$$\Lambda(t) = \frac{p(t+1) - p(t)}{p(t)} = e^{\eta(t)} - 1, \quad (2.4)$$

which is referred to as return per unit capital of the asset. The properties of a random multiplicative process are expressed in terms of the average return per

capital $\alpha = \langle \Lambda \rangle = \langle e^\eta \rangle - 1$, and the volatility (standard deviation) of the return per capital $\sigma = \sqrt{\langle \Lambda^2 \rangle - \langle \Lambda \rangle^2}$. In our notation $\alpha = \langle e^\eta \rangle - 1$ is determined by the *average* and not typical growth rate of the process. For $\eta \ll 1$, $\alpha \simeq v + D/2 + v^2/2$, while the volatility σ is related to D (the dispersion of η) through $\sigma \simeq \sqrt{D}$.

Expanding Eq. (2.3) up to the second order in $\Lambda = e^\eta - 1$ one gets: $v_{\text{typ}} \simeq \langle r(e^\eta - 1) - r^2(e^\eta - 1)^2 \rangle = \alpha r - (\sigma^2 + \alpha^2)r^2/2$. The optimal r is given by

$$r_{\text{opt}} = \frac{\alpha}{\alpha^2 + \sigma^2}. \quad (2.5)$$

If the above formula prescribes $r_{\text{opt}} > 1$, the investor is advised to trust his whole capital to this asset. We remind you that in this paper the risk-free return per capital is set to zero (investor keeps the rest of his capital in cash). In a more realistic case, when a risk-free bank deposit brings a return p during a single readjustment interval, the formula for the optimal investment ratio should be generalized to

$$r_{\text{opt}} = \frac{\alpha - p}{\alpha^2 + \sigma^2}. \quad (2.6)$$

In a hypothetical case discussed by Merton [3], when asset's price follows a continuous multiplicative random walk (i.e. price fluctuations are uncorrelated at the smallest time scale) and the investor is committed to adjust his investment ratio on a continuous basis, one should use infinitesimal quantities $\alpha \rightarrow \alpha dt$ and $\sigma^2 \rightarrow \sigma^2 dt$. Under these circumstances the term $\alpha^2 dt^2$, being second order in infinitesimal time increment dt , should be dropped from the denominator. Then one recovers an optimal investment fraction for "logarithmic utility" derived by Merton [3].

Asset price fluctuations encountered in developed financial markets have relatively large average returns and small volatilities, so that the optimal investment fraction into any given asset r_i^{opt} is almost always bigger than 1. For instance the data for average annual return and volatility of Dow-Jones index in 1954-1963 [10] are $\alpha_{DJ} = 16\%$, $\sigma_{DJ} = 20\%$, while the average risk-free interest rate is $p = 3\%$. This suggests that for an investor committed to yearly readjustment of his asset holdings to the selected ratio, the optimal investment ratio in Dow-Jones portfolio is $r_{DJ} = (\alpha_{DJ} - p)/(\sigma_{DJ}^2 + \alpha_{DJ}^2) = 1.98 > 1$. On the other hand the investor ready to readjust his stock holdings every month should use $\alpha_{\text{monthly}} \simeq \alpha/12$ and $\sigma_{\text{monthly}} \simeq \sigma/\sqrt{12}$. For him the optimal investment fraction would be $r_{DJ}^{\text{monthly}} = (\alpha_{DJ}/12 - p/12)/(\sigma_{DJ}^2/12 + (\alpha_{DJ}/12)^2) \simeq 3.09$. In both cases, given no other alternatives the investor interested in a long-term capital growth is advised to trust his whole capital to Dow-Jones portfolio and enjoy a *typical* annual return $\alpha - \sigma^2/2 = 14\%$, which is 2% smaller than the average annual return of 16% but significantly bigger than the risk-free return of 3%.

3. Optimization of Multi-Asset Portfolio

We proceed by generalizing the results of a previous section to a more realistic situation, where the investor can keep a fraction of his total capital in a portfolio

composed of N risky assets. The returns per unit capital of different assets are defined as $\Lambda_i(t) = \frac{p_i(t+1) - p_i(t)}{p_i(t)} = e^{\eta_i} - 1$. Each asset is characterized by an average return per capital $\alpha_i = \langle e^{\eta_i} \rangle - 1$, and volatility $\sigma_i = \sqrt{\langle e^{2\eta_i} \rangle - \langle e^{\eta_i} \rangle^2}$. As in the single asset case, an investor has decided to maintain a given fraction r_i of his capital invested in i th asset, and to keep the rest in cash. His goal is to maximize the *typical* growth rate of his capital by selecting the optimal set of r_i . The explicit expression for the typical rate under those circumstances is given by

$$v_{\text{typ}}(r_1, r_2 \dots r_N) = \left\langle \ln \left[1 + \sum_{i=1}^N r_i (e^{\eta_i} - 1) \right] \right\rangle. \quad (3.7)$$

The task of finding an analytical solution for the global maximum of this expression seems hopeless. We can, however, expand the logarithm in Eq. (3.7), assuming that all returns $\Lambda_i = e^{\eta_i} - 1$ are small. Then to a second order one gets: $v_{\text{typ}} = \sum_i \alpha_i r_i - \sum_{i,j} K_{ij} r_i r_j / 2$, where K_{ij} is a covariance matrix of returns, defined by $K_{ij} = \langle \Lambda_i \Lambda_j \rangle$. In this work we restrict ourselves to the case of uncorrelated assets, when the only non-zero elements of covariance matrix lay on the diagonal, $K_{ij} = (\alpha_i^2 + \sigma_i^2) \delta_{ij}$. In this case the expression for typical rate becomes

$$v_{\text{typ}} = \sum_{i=1}^N [\alpha_i r_i - (\sigma_i^2 + \alpha_i^2) r_i^2 / 2], \quad (3.8)$$

without any restrictions the optimal investment fraction in a given asset is determined by a single asset formula (2.5)

$$\tilde{r}_i^{\text{opt}} = \frac{\alpha_i}{\sigma_i^2 + \alpha_i^2}. \quad (3.9)$$

In case of the general covariance matrix the above formula becomes

$$\tilde{r}_i^{\text{opt}} = \sum_j (K^{-1})_{ij} \alpha_j, \quad (3.10)$$

where $(K^{-1})_{ij}$ is an element of a matrix inverse to K_{ij} . With somewhat heavier algebra all results from the following paragraphs can be reformulated to include the effects of a general covariance matrix and non-zero risk-free interest rate. However, we will not attempt it in this paper.

The non-trivial part of the N asset case comes from the restriction $\sum r_i \leq 1$, $r_i \geq 0$. This restriction starts to be relevant if $\sum_i \tilde{r}_i^{\text{opt}} > 1$, and the Eq. (3.9) no longer works. In this case the optimal solution would be to invest the whole capital in assets and to search for a maximum of v_{typ} restricted to the hyperplane $\sum_i r_i = 1$. This interesting case was not considered by Merton [3]. Therefore, his prescription for the vector of optimal investment fractions holds only when $\sum \alpha_i / \sigma_i^2 \leq 1$. Introducing a Lagrange multiplier λ , one gets $r_i^{\text{opt}} = (\alpha_i - \lambda) / (\alpha_i^2 +$

σ_i^2). Obviously, the assets for which $r_i < 0$ should be dropped and the optimal r_i^{opt} are finally given by

$$r_i^{\text{opt}} = \frac{\alpha_i - \lambda}{\alpha_i^2 + \sigma_i^2} \theta \left(\frac{\alpha_i - \lambda}{\alpha_i^2 + \sigma_i^2} \right), \quad (3.11)$$

where $\theta(x)$ is a usual Heavyside step function. The Lagrange multiplier λ is found by solving

$$\sum_{i=1}^N \frac{\alpha_i - \lambda}{\alpha_i^2 + \sigma_i^2} \theta \left(\frac{\alpha_i - \lambda}{\alpha_i^2 + \sigma_i^2} \right) = 1. \quad (3.12)$$

To demonstrate how this optimization works in practice we consider the following simple example. An investor has an alternative to invest his capital in three assets with average returns $\alpha_1 = 1.5\%$, $\alpha_2 = 2\%$, $\alpha_3 = 2.5\%$. Each of these assets has the same volatility $\sigma = 10\%$. Which are optimal investment fractions in this case? Equation (3.9) recommends $\tilde{r}_1^{\text{opt}} = \alpha_1/(\sigma^2 + \alpha_1^2) \simeq \alpha_1/\sigma^2 = 1.5$, $\tilde{r}_2^{\text{opt}} \simeq 2$, $\tilde{r}_3^{\text{opt}} \simeq 2.5$. Each of these numbers is bigger than one, which means that given any one of these assets as the only investment alternative, the investor would be advised to trust his whole capital to it. As explained above, whenever Eq. (3.9) results in $\tilde{r}_1^{\text{opt}} + \tilde{r}_2^{\text{opt}} + \tilde{r}_3^{\text{opt}} > 1$, the investor should not keep any money in cash. We need to solve Eq. (3.11) to determine how he should share his capital between three available assets. Assuming first that each asset gets a non-zero fraction of the capital, one writes Eq. (3.12) for the Lagrange multiplier λ : $1.5 - \lambda + 2 - \lambda + 2.5 - \lambda = 1$, or $\lambda = 5/3 \simeq 1.67$. But then $r_1 = 1.5 - \lambda$ is negative. This suggests that the average return in asset 1 is too small, and that the whole capital should be divided between assets 2 and 3. Then Eq. (3.12) $2 - \lambda + 2.5 - \lambda = 1$ has the solution $\lambda = 1.75$, and the optimal investment fractions are $r_1^{\text{opt}} = 0$, $r_2^{\text{opt}} = 0.25$, $r_3^{\text{opt}} = 0.75$. This optimum represents the compromise between the following two tendencies. On the one hand, diversification of the portfolio tends to increase its typical growth rate and bring it closer to the average growth rate. This happens because fluctuations of different asset's prices partially cancel each other making the whole portfolio less risky. On the other hand, to diversify the portfolio one has to use assets with α 's smaller than that of the best asset in the group, and thus compromise the average growth rate itself. In the above example the average return α_1 was just too low to justify including it in the portfolio.

Finally, we want to compare our results with the exact formula derived by Breiman [8]. His argument goes as follows: in case where there is no bank (or it is just included as the alternative of investing in a risk-free asset for which $\Lambda = p$ and $\sigma = 0$) one wants to maximize $\langle \ln \sum r_i e^{\eta_i} \rangle$ subject to the constraint $\sum r_i = 1$. Introducing a Lagrange multiplier β (different from Lagrange multiplier λ used above) one gets a condition for an extremal value of growth rate: $\langle e^{\eta_i} / \sum r_i e^{\eta_i} \rangle - \beta = 0$. This can also be written as $\langle r_i e^{\eta_i} / \sum r_i e^{\eta_i} \rangle - \beta r_i = 0$. The summation over i shows that $\beta = 1$, therefore at optimum is determined by a solution of the system of N

equations:

$$r_i = \left\langle r_i e^{\eta_i} / \sum_j r_j e^{\eta_j} \right\rangle. \quad (3.13)$$

Notice that the i th equation automatically holds if $r_i = 0$. Therefore, finding an optimal set of investment fractions r_i is equivalent to solving (3.13) with $r_i \geq 0$. According to this equation in the strategy, optimal in Kelly's sense, *on average* one does not have to buy or sell assets since the *average* fraction of each asset's capital in the total capital ($\langle r_i e^{\eta_i} / \sum_j r_j e^{\eta_j} \rangle$) is conserved by dynamics. Unfortunately, the exact set of Eq. (3.13) is as unusable as it is elegant: it suggests no constructive way to derive the set of optimal investment fractions from known asset's average returns and covariance matrix. In this sense our set of approximate Eq. (3.11) provides an investor with a constructive method to iteratively determine the set of optimal weights of different assets in the optimal portfolio.

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